THE INITIAL PRESSURE ON HALF-SPACE UNDER THE IMPACT OF A SMOOTH FLAT DIE

(O NACHAL'NOM DAVLENII NA POLUPROSTRANSTVO GLADKOGO PLOSKOGO SHTAMPA PRI UDARE)

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The paper applies the well-known results obtained by Naryshkina to the evaluation of the initial stresses on the area of contact between elastic bodies in the absence of total friction.

1. If we replace the two bodies by half-spaces $z_j \geqslant 0$ subjected only to normal stresses on the area of contact, we obtain following Expression for the displacements of points in the directions of the axes OZ_j [1] :

$$\int_0^{t_0} W_j(x_0, y_0, z_{j0}, t)\,dt = -\frac{1}{8\pi\rho_j}\left(\frac{\partial M^j}{\partial z_{j0}} + \frac{\partial N_3^j}{\partial x_0} - \frac{\partial N_4^j}{\partial y_0}\right) \tag{1.1}$$

where M^j for zero initial values and body forces have the form

$$M^j = \iiint_{T_j} W_{j1}{}^\circ \sigma_z\,dT \qquad (j = 1, 2) \tag{1.2}$$

The domain T_j, which is three-dimensional, belongs to the hyperplane $z_j = 0$; the displacements of points $W_{j1}{}^\circ$ in the direction of these axes are determined by the fundamental solutions of the longitudinal type; the quantities $N_3{}^j$ and $N_4{}^j$ may be obtained from (1.2) by replacing $W_{j1}{}^\circ$ by the appropriate displacements from the fundamental solutions of the transverse type.

The principal part of $W_{j1}{}^\circ$ is given by

$$W_{j1}{}' = \frac{z_j - z_{j0}}{r_j^3}\left[(t_0 - t)^2 - r_j^2 a_j^{-2}\right] = -\frac{1}{2\pi}\,\mathrm{Re}\int_0^{2\pi} \theta_{j1}{}^2\,d\lambda$$
$$r_j^2 = (x - x_0)^2 + (y - y_0)^2 + (z_j - z_{j0})^2 \tag{1.3}$$

Here a_j are the velocities of propagation of longitudinal waves in the media. From now on we shall denote the velocities of propagation of transverse waves by b_j. The functions θ_{j1} are given by the relations

$$t_0 - t - \theta_{j1}\rho'\cos(\varphi - \lambda) + (z_j - z_{j0})\,L_{j1}(\theta_{j1}) = 0$$
$$L_{j1}(\theta) = \sqrt{a_j^{-2} - \theta^2}, \qquad x - x_0 = \rho'\cos\varphi, \qquad y - y_0 = \rho'\sin\varphi \tag{1.4}$$

The remaining parts, which correspond to the reflected waves, may be found from Formulas

$$W_{j1}'' = -\frac{1}{2\pi}\,\mathrm{Re}\int_0^{2\pi}\int_0^{\theta_{j1}''}\frac{F_j^-(\theta)}{F_j^+(\theta)}\,\Psi'(\theta)\,d\theta\,d\lambda, \qquad \Psi(\theta) = \theta^2, \qquad \Psi' = 2\theta$$

$$W_{j1}''' = +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\theta_{j1}'''}\frac{4\theta^2(2\theta^2 - b_j^{-2})}{F^+(\theta)}\,\Psi'(\theta)\,d\theta\,d\lambda$$

$$F_1^{\pm}(\theta) = (b_j^2 - 2\theta^2)^2 \pm 4\theta^2 L_{j1}(\theta)\,L_{j2}(\theta), \qquad L_{j2}(\theta) = \sqrt{b_j^{-2} - \theta^2} \tag{1.5}$$

The principal parts of the fundamental solutions W_{j3}° and W_{j4}° are

$$W_{j3}' = -\frac{x - x_0}{r_j^3}\left[(t_0 - t)^2 - r_j^2 b_j^{-2}\right] = -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,\chi(\theta)\cos\lambda\,d\lambda$$

$$W_{j4}' = +\frac{y - y_0}{r_j^3}\left[(t_0 - t)^2 - r_j^2 b_j^{-2}\right] = +\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,\chi(\theta)\sin\lambda\,d\lambda \tag{1.6}$$

$$\chi(\theta) = \theta L_{j2}(\theta) - b_j^{-2}\sin^{-1} b_j\theta$$

To them corresponds the reflected disturbance

$$W_{j3}'' = -\frac{1}{2\pi}\,\mathrm{Re}\int_0^{2\pi}\int_0^{\theta_{j3}''}\frac{4(2\theta^2 - b_j^{-2})\,L_{j1}(\theta)\,L_{j2}(\theta)}{F_j^+(\theta)}\,\chi'(\theta)\cos\lambda\,d\theta\,d\lambda$$

$$W_{j4}'' = -\frac{1}{2\pi}\,\mathrm{Re}\int_0^{2\pi}\int_0^{\theta_{j4}''}\frac{4(2\theta^2 - b_j^{-2})\,L_{j1}(\theta)\,L_{j2}(\theta)}{F_j^+(\theta)}\,\chi'(\theta)\sin\lambda\,d\theta\,d\lambda. \tag{1.7}$$

$$W_{j3}''' = +\frac{+1}{2\pi}\,\mathrm{Re}\int_0^{2\pi}\int_0^{\theta_{j3}'''}\frac{F_j^-(\theta)}{F_j^+(\theta)}\,\chi'(\theta)\cos\lambda\,d\theta\,d\lambda$$

$$W_{j4}''' = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\theta_{j4}'''}\frac{F_j^-(\theta)}{F_j^+(\theta)}\,\chi'(\theta)\sin\lambda\,d\theta\,d\lambda, \qquad \chi'(\theta) = \frac{d\chi}{d\theta} \tag{1.8}$$

The variables θ_{jk}'' and θ_{jk}''' may be determined from equalities of the type (1.4). Making use of the properties of the fundamental solutions we may easily justify the insertion of the differential sign under the integral in (1.1). We then obtain

$$\int_0^{t_0} W_j(x_0, y_0, z_{j0}, t)\,dt = -\frac{1}{8\pi\rho_j}\iiint_{T_j}\left(\frac{\partial W_{j1}^{\circ}}{\partial z_{j0}} + \frac{\partial W_{j3}^{\circ}}{\partial x_0} - \frac{\partial W_{j4}^{\circ}}{\partial y_0}\right)\sigma_z\,dT \tag{1.9}$$

Letting z_{j0} tend to zero we find that

$$\int_0^{t_0} W_j(x_0, y_0, 0, t)\,dt = \frac{1}{4\pi^2 b_j^4\rho_j}\iiint_{T_0}\mathrm{Re}\int_0^{2\pi}\frac{\theta L_{j1}(\theta)\,d\lambda}{F_j^+(\theta)\,\rho\cos(\varphi - \lambda)}\,P(x, y, t)\,dT \tag{1.10}$$

where ρ_j are the densities of the media

$$\theta = \frac{t_0 - t}{\rho'\cos(\varphi - \lambda)}, \qquad \rho' = (x - x_0)^2 + (y - y_0)^2, \qquad P(x, y, t) = -\sigma_z(x, y, t)$$

Adding, we obtain the relative displacement of the bodies

$$\int_0^{t_0} W(x_0, y_0, 0, t)\, dt =$$

$$= \frac{1}{4\pi^2} \sum_{j=1}^{2} \frac{1}{\rho_j b_j^4} \int_0^{t_0} \left\{ \iint_B \left[\mathrm{Re} \int_0^{2\pi} \frac{\theta L_{j0}(\theta)\, d\lambda}{F_j^+(\theta)\, \rho' \cos(\varphi - \lambda)} \right] P(x, y, t)\, dx dy \right\} dt \qquad (1.11)$$

where B, an area in the xy plane, is the common part of the area of contact and of the base of the circular cone with apex at the point (x_0, y_0, t_0) to which the hypercone degenerates at $z_j = 0$ and $z_{j0} = 0$. Suppose that t_0 is so small that the base of this cone lies entirely within the area of contact, then B will be a circle with center at the point $(x_0, y_0, 0)$ and radius $a_j(t_0 - t)$. Expanding $P(x, y, t)$ into a series in the variables x, y, t in the vicinity of the point $(x_0, y_0, 0)$, we obtain

$$\int_0^{t_0} W(x_0, y_0, 0, t)\, dt =$$

$$= \frac{P(x_0, y_0, 0)}{4\pi^2} \sum_{j=1}^{2} \frac{1}{\rho_j b_j^4} \int_0^{t_0}\int_0^{2\pi}\int_0^{a_j(t_0-t)} \mathrm{Re} \int_0^{2\pi} \frac{\theta L_{j1}(\theta)\, d\lambda}{F_j^+(\theta) \cos(\varphi - \lambda)} d\rho' d\varphi dt + \ldots \qquad (1.12)$$

where terms that do not affect the final result have been omitted. Setting

$$\xi = \frac{\rho' \cos(\varphi - \lambda)}{t_0 - t}, \qquad \alpha_j = a_j \cos(\varphi - \lambda)$$

we find

$$\int_0^{t_0}\int_0^{2\pi}\int_0^{a_j(t_0-t)} \mathrm{Re} \int_0^{2\pi} \frac{\theta L_{j1}(\theta)\, d\lambda}{F_j^+(\theta) \cos(\varphi - \lambda)} d\rho' d\varphi dt = \frac{t_0^2}{2} \int_0^{2\pi}\int_0^{2\pi}\int_0^{\alpha_j} \mathrm{Re} \frac{L_{j1}(\xi^{-1})\, d\xi}{F_j^+(\xi^{-1})\, \xi} \frac{d\lambda d\varphi}{\cos^2(\lambda - \varphi)} \qquad (1.13)$$

Let us evaluate the integral of the right-hand side. Denoting the integrand by Φ we can write

$$K_j^\circ = \int_0^{2\pi}\int_0^{2\pi} \Phi\, [\cos(\varphi - \lambda)]\, d\lambda d\varphi \qquad (1.14)$$

but

$$\frac{\partial}{\partial \varphi} \int_0^{2\pi} \Phi\, [\cos(\varphi - \lambda)]\, d\lambda = \frac{\partial}{\partial \varphi} \int_{-\varphi}^{2\pi - \varphi} \Phi(\cos \lambda)\, d\lambda = 0 \qquad (1.15)$$

Therefore

$$K_j^\circ = 2\pi \int_0^{2\pi} \Phi\, [\cos(\varphi - \lambda)]\, d\lambda = 2\pi \int_{\pi}^{\pi} \Phi(\cos \lambda)\, d\lambda = 4\pi \int_0^{\pi} \Phi(\cos \lambda)\, d\lambda \qquad (1.16)$$

Taking account of the choice of the branches of the radicals and their value, we obtain

$$\int_0^\pi \Phi d\lambda = \int_0^{\varkappa_1}\int_{b_j}^{\alpha_1} \operatorname{Re}\frac{-i\sqrt{\xi^{-2}-a_j^{-2}}}{F_j^+(\xi^{-1})\,\xi}\,d\xi\,\frac{d\lambda}{\cos^2\lambda} + \int_{\varkappa_2}^{1/_2\pi}\int_{b_j}^{\alpha_2} \operatorname{Re}\frac{i\sqrt{\xi^{-2}-a_j^{-2}}}{F_j^+(\xi^{-1})\,\xi}\,d\xi\,\frac{d\lambda}{\sin^2\lambda} \qquad (1.17)$$

$$(\varkappa_1 = \cos^{-1}\sqrt{k_j},\ \varkappa_2 = \sin^{-1}\sqrt{k_j},\ k_j = b_j^2 / a_j^2,\ \alpha_1 = a_j\cos\lambda,\ \alpha_2 = a_j \sin\lambda)$$

Carrying out both integrals by parts and making a change of variables, putting $\eta = \tan\lambda$ in the first and $\eta = \cot\lambda$ in the second, we obtain

$$\int_0^\pi \Phi(\cos\lambda)\,d\lambda = 2a_j^3\int_0^{\eta_{0j}} \frac{4\sqrt{\eta_{0j}^2-\eta^2}\,\eta^4\,d\eta}{[\eta_{0j}^2-1-2\eta^2]^4+16(1+\eta^2)(\eta_{0j}^2-\eta^2)\,\eta^2} \qquad \left(\eta_{0j} = \frac{1}{k_j}-1\right) \qquad (1.18)$$

We adopt the notation

$$Q(k_j) = k_j^{-3/2}\int_0^{\eta_{0j}} \frac{\sqrt{\eta_{0j}^2-\eta^2}\,\eta^4 d\eta}{[\eta_{0j}^2-1-2\eta^2]^4+16(1+\eta^2)(\eta_{0j}^2-\eta^2)\,\eta^2}. \qquad (1.19)$$

This integral may be evaluated in elementary functions. For example, if $k_j = 1/3$, then $Q(1/_3) \approx 0.0717$. Collecting results, we obtain

$$K_j^\circ = 24\pi a_j^3 k_j^{3/2}\, Q(k_j) \qquad (1.20)$$

Returning now to equality (1.12), we write this in the form

$$\int_0^{t_0} W(x_0, y_0, 0, t)\,dt = t_0^2\,\frac{4P(x_0, y_0, 0)}{\pi}\sum_{j=1}^{2}\frac{Q(k_j)}{\rho_j b_j} + \ldots \qquad (1.21)$$

The terms omitted are of the order of $t^3{}_0$ and higher. Differentiating twice with respect to t and letting $t_0 \to 0$, we arrive at the relation

$$\left(\frac{\partial W}{\partial t_0}\right)_{t_0=0} = \frac{8P(x, y, 0)}{\pi}\sum_{j=1}^{2}\frac{Q(k_j)}{\rho_j b_j} \qquad (1.22)$$

which for a given function W enables us to find the initial value of the normal stress at points on the area of contact between the bodies.

2. If one of the bodies, for example, the first, is rigid (a die with a flat base) the result (1.22) is simplified

$$b_1 = \infty, \qquad P(x_0, y_0, 0) = \frac{\pi}{8}\,\frac{\rho b}{Q(k)}\,V_0 \qquad (2.1)$$

where V_0 is the relative velocity of impact, and ρ, b and k refer to the second body. Thus the stresses P are uniform at all points under the die. This enables us to calculate the force exerted by the die on the half-space

$$PS = \frac{\pi}{8}\,\frac{\rho b}{Q(k)}\,V_0 S \qquad (2.2)$$

where S is the area of the base of the die. Thus, upon impact of a die on a half-space the force increases spasmodically, reaching instantaneously the value (2.2). When initial contact occurs at points or on lines the above force is zero since in this case $S = 0$.

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